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The 't Hooft six quark flavor mixing interaction ($N_f = 3$) is bosonized by the path integral method. The considered complete Lagrangian is constructed on the basis of the combined 't Hooft and $U(3) \times U(3)$ extended chiral four fermion Nambu – Jona-Lasinio interactions. The method of the steepest descents is used to derive the effective mesonic Lagrangian. Additionally to the known lowest order stationary phase (SP) result of Reinhardt and Alkofer we obtain the contribution from the small quantum fluctuations of bosonic configurations around their stationary phase trajectories. Fluctuations give rise to multivalued solutions of the gap equations, marked at instances by drastic changes in the quark condensates, a novel scenario for the vacuum state of hadrons at low energies.

I. INTRODUCTION

The global $U_L(3) \times U_R(3)$ chiral symmetry of the QCD Lagrangian (for massless quarks) is broken by the $U_A(1)$ Adler-Bell-Jackiw anomaly of the $SU(3)$ singlet axial current $\bar{q}\gamma_\mu\gamma_5 q$. Through the study of instantons [1,2], it became clear that effective $2N_f$ quark interactions, known as 't Hooft interactions, violate $U_A(1)$, but are still invariant under chiral $SU(N_f) \times SU(N_f)$.In the case of two flavors they are four-fermion interactions, and the resulting low-energy theory resembles the old Vaks Larkin Nambu Jona-Lasinio model [3]. In the case of three flavors they are six-fermion interactions which are responsible for the correct description of η and η' physics, and additionally lead to the OZI-violating effects [4,5],

$$\mathcal{L}_{\rm det} = \kappa(\det\bar{q}P_R q + \det\bar{q}P_L q) \qquad (1)$$

where the matrices $P_{R,L} = (1 \pm \gamma_5)/2$ are projectors and the determinant is over flavor indices.

The physical degrees of freedom of QCD at low-energies are mesons. The bosonization of the effective quark interaction (1) by the path integral approach has been considered in [6], where the lowest order stationary phase approximation (SPA) has been used to estimate the leading contribution from the 't Hooft determinant. In this approximation the functional integral is dominated by the stationary trajectories $r_{\rm st}(x)$, determined by the extremum condition $\delta S(r) = 0$ of the action $S(r)$. The lowest order SPA corresponds to the case in which the integrals associated with $\delta^2 S(r)$, for the path $r_{\rm st}(x)$ are neglected and only $S(r_{\rm st})$ contributes to the generating functional. The subject of the present work is to include the contribution associated with $\delta^2 S(r)$. Our motivation is the following: let us consider the one-dimensional integrals of the form

$$F(\lambda) = \int_{-\infty}^{\infty} dt \exp[i\lambda f(t)] \qquad (2)$$

What is sought by the method of SP is the dominant contribution to F as $\lambda \to \infty$. The dominant contribution to the integral comes from regions of t where f' vanishes. Supposing that f' vanishes at only one point t_0 and neglecting contributions to the integral from regions far from t_0, one can obtain the result

$$F(\lambda) = \sqrt{\frac{2\pi i}{\lambda f''(t_0)}}\, e^{i\lambda f(t_0)}. \qquad (3)$$

Hence $F(\lambda)$ goes to zero like $1/\sqrt{\lambda}$. The lowest order SPA would correspond to the result $F(\lambda) = const \cdot \exp[i\lambda f(t_0)]$, which has the incorrect asymptotic behavior $F(\lambda) = \mathcal{O}(1)$. If results related to finite-dimensional integrals, such as $F(\lambda)$, mean anything with regard to corresponding functional integrals, one can conclude that the leading SPA should include the contribution from the integrals associated with the second functional derivative $\delta^2 S(r_{\rm st})$.

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II. PATH INTEGRAL BOSONIZATION

To be definite, let us consider the theory of the quark fields in four dimensional Minkowski space, with dynamics defined by the Lagrangian density

$$\mathcal{L} = \mathcal{L}_{\rm NJL} + \mathcal{L}_{\rm det}. \tag{4}$$

The first term here is the extended version of the Nambu – Jona-Lasinio (NJL) Lagrangian $\mathcal{L}_{\rm NJL} = \mathcal{L}_0 + \mathcal{L}_{\rm int}$, consisting of the free field part

$$\mathcal{L}_0 = \bar{q}(i\gamma^\mu\partial_\mu - \hat{m})q, \tag{5}$$

and the $U(3)_L \times U(3)_R$ chiral symmetric four-quark interaction

$$\mathcal{L}_{\rm int} = \frac{G}{2}[(\bar{q}\lambda_a q)^2 + (\bar{q}i\gamma_5\lambda_a q)^2]. \tag{6}$$

We assume that quark fields have color and flavor indices running through the set $i = 1, 2, 3$; λ_a are the standard $U(3)$ Gell-Mann matrices with $a = 0, 1, \ldots, 8$. The current quark mass, $\hat{m}$, is a nondegenerate diagonal matrix with elements $\mathrm{diag}(\hat{m}_u, \hat{m}_d, \hat{m}_s)$, it explicitly breaks the global chiral $U(3)_L \times U(3)_R$ symmetry of the $\mathcal{L}_{\rm NJL}$ Lagrangian. The second term in (4) is given by (1). Letting

$$s_a = -\bar{q}\lambda_a q, \quad p_a = \bar{q}i\gamma_5\lambda_a q, \quad s = s_a\lambda_a, \quad p = p_a\lambda_a \tag{7}$$

yields

$$\mathcal{L}_{\rm det} = -\frac{\kappa}{64}\left[\det(s + ip) + \det(s - ip)\right]. \tag{8}$$

The dynamics of the system is described by the vacuum transition amplitude in the form of the path integral

$$Z = \int \mathcal{D}q\mathcal{D}\bar{q}\exp\left(i\int d^4x\mathcal{L}\right). \tag{9}$$

which can be written in the equivalent form [6]

$$Z = \int \mathcal{D}\sigma_a\mathcal{D}\phi_a\mathcal{D}q\mathcal{D}\bar{q}\exp\left(i\int d^4x\mathcal{L}_q\right)\int \mathcal{D}r_{1a}\mathcal{D}r_{2a}\exp\left(i\int d^4x\mathcal{L}_r\right) \tag{10}$$

where

$$\mathcal{L}_q(\bar{q}, q, \sigma, \phi) = \bar{q}(i\gamma^\mu\partial_\mu - \hat{m} - \sigma + i\gamma_5\phi)q, \tag{11}$$

$$\begin{aligned}\mathcal{L}_r(\sigma, \phi, r_1, r_2) = {} & 2G\left[(r_{1a})^2 + (r_{2a})^2\right] - 2(r_{1a}\sigma_a + r_{2a}\phi_a) \\ & - \frac{\kappa}{8}\left[\det(r_1 + ir_2) + \det(r_1 - ir_2)\right].\end{aligned} \tag{12}$$

The Fermi fields enter the action bilinearly, we can always integrate over them. At this stage one should also shift the scalar fields $\sigma_a \to \sigma_a + \Delta_a$ by demanding that the vacuum expectation values of the shifted fields vanish $< 0|\sigma_a|0 >= 0$, yielding gap equations to fix parameters $\Delta_a = m_a - \hat{m}_a$, where m_a denotes the constituent quark masses [1]. To evaluate path integrals over $r_{1,2}$ one has to use the method of stationary phase, or, after the formal analytic continuation in the time coordinate $x_4 = ix_0$, the method of steepest descents. The analytical continuation of the Euclidean version of the path integral under consideration is (see [8] for details),

$$J(\sigma, \phi) = \int_{-i\infty + r_{\rm st}}^{+i\infty + r_{\rm st}} \mathcal{D}r_{1a}\mathcal{D}r_{2a}\exp\left(\int d^4x\mathcal{L}_r(\sigma, \phi, r_1, r_2)\right). \tag{13}$$

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[1]The shift by the current quark mass is needed to hit the correct vacuum state, see e.g. [7].

Near the saddle point $r^a_{\rm st}$,

$$\mathcal{L}_r \approx \mathcal{L}_r(r_{\rm st}) + \frac{1}{2}\sum_{\alpha,\beta} \tilde{r}_\alpha \mathcal{L}''_{\alpha\beta}(r_{\rm st})\tilde{r}_\beta \tag{14}$$

where $r^a_{\rm st}$ is a solution of the equations $\mathcal{L}'_r(r_1, r_2) = 0$ determining a flat spot of the surface $\mathcal{L}_r(r_1, r_2)$,

$$\left\{ \begin{array}{rcl} 2Gr^a_1 - (\sigma + \Delta)_a - \frac{3\kappa}{8} A_{abc}(r^b_1 r^c_1 - r^b_2 r^c_2) &=& 0 \\ 2Gr^a_2 - \phi_a + \frac{3\kappa}{4} A_{abc} r^b_1 r^c_2 &=& 0. \end{array} \right. \tag{15}$$

with totally symmetric constants, A_{abc}, closely related with the $U(3)$ constants d_{abc}. We use in (14) symbols $\tilde{r}^a$ for the differences $(r^a - r^a_{\rm st})$. To deal with the multitude of integrals in (13) we define a column vector $\tilde{r}$ with eighteen components $\tilde{r}_\alpha = (\tilde{r}^a_1, \tilde{r}^a_2)$ and with the matrix $\mathcal{L}''_{\alpha\beta}(r_{\rm st})$ being equal to

$$\mathcal{L}''_{\alpha\beta}(r_{\rm st}) = 4GQ_{\alpha\beta}, \quad Q_{\alpha\beta} = \left(\begin{array}{cc} \delta_{ab} - \frac{3\kappa}{8G} A_{abc} r^c_{1\rm st} & \frac{3\kappa}{8G} A_{abc} r^c_{2\rm st} \\ \frac{3\kappa}{8G} A_{abc} r^c_{2\rm st} & \delta_{ab} + \frac{3\kappa}{8G} A_{abc} r^c_{1\rm st} \end{array} \right). \tag{16}$$

Eq.(13) can now be concisely written as

$$J(\sigma,\phi) = \exp\left(\int d^4x \mathcal{L}_r(r_{\rm st}) \right) \int_{-i\infty}^{+i\infty} \mathcal{D}\tilde{r}_\alpha \exp\left(2G \int d^4x \tilde{r}^{\rm t} Q(r_{\rm st}) \tilde{r} \right) [1 + \mathcal{O}(\hbar)]. \tag{17}$$

The solutions of Eqs.(15) are the following even and odd parity combinations $r^a_{1\rm st}$ and $r^a_{2\rm st}$ expressed in the form of increasing powers in σ_a, ϕ_a

$$r^a_{1\rm st} = h_a + h^{(1)}_{ab}\sigma_b + h^{(1)}_{abc}\sigma_b\sigma_c + h^{(2)}_{abc}\phi_b\phi_c + \ldots \tag{18}$$

$$r^a_{2\rm st} = h^{(2)}_{ab}\phi_b + h^{(3)}_{abc}\phi_b\sigma_c + \ldots \tag{19}$$

Putting these expansions in Eqs.(15) one obtains a series of selfconsistent equations to determine the constants h_a, $h^{(1)}_{ab}$, $h^{(2)}_{ab}$, etc. [8] As a result we get

$$\mathcal{L}_r(r_{\rm st}) = -2h_a\sigma_a - h^{(1)}_{ab}\sigma_a\sigma_b - h^{(2)}_{ab}\phi_a\phi_b + \mathcal{O}(\text{field}^3). \tag{20}$$

We now turn to the evaluation of the path integral in Eq.(17). In order to define the measure $\mathcal{D}\tilde{r}_\alpha$ more accurately we expand $\tilde{r}_\alpha$ in a Fourier series

$$\tilde{r}_\alpha(x) = \sum_{n=1}^{\infty} c_{n,\alpha}\varphi_n(x), \tag{21}$$

assuming that suitable boundary conditions are imposed. The set of the real functions $\{\varphi_n(x)\}$ form an orthonormal and complete sequence, therefore

$$\int \mathcal{D}\tilde{r}_\alpha \exp\left(2G \int d^4x \tilde{r}^{\rm t} Q(r_{\rm st}) \tilde{r} \right) = \frac{C}{\sqrt{\det(2G\lambda^{\alpha\beta}_{nm})}}. \tag{22}$$

with

$$2G\lambda^{\alpha\beta}_{nm} = \left(\begin{array}{cc} h^{(1)-1}_{ac} & 0 \\ 0 & h^{(2)-1}_{ac} \end{array} \right)_{\alpha\sigma} \left(\delta_{\sigma\beta}\delta_{nm} + \int d^4x \varphi_n(x) F_{\sigma\beta}(x) \varphi_m(x) \right) \tag{23}$$

and

$$F_{\sigma\beta} = \frac{3\kappa}{4} A_{eba} \left(\begin{array}{cc} -h^{(1)}_{ce}(r^a_{1\rm st} - h_a) & h^{(1)}_{ce} r^a_{2\rm st} \\ h^{(2)}_{ce} r^a_{2\rm st} & h^{(2)}_{ce}(r^a_{1\rm st} - h_a) \end{array} \right)_{\sigma\beta}. \tag{24}$$

Only the matrix $F_{\sigma\beta}$ depends here on fields σ, ϕ. By absorbing in C the irrelevant field independent part of $2G\lambda^{\alpha\beta}_{nm}$, and expanding the logarithm in the representation $\det(1+F) = \exp \operatorname{tr}\ln(1+F)$, one can obtain finally for the complete action

$$S_r = \int d^4x \left\{ \mathcal{L}_r(r_{\rm st}) + \frac{a}{2G^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \operatorname{tr}\left[F^n_{\alpha\beta}(r_{\rm st}) \right] \right\} \tag{25}$$

proposing that the undetermined dimensionless constant a will be fixed by confronting the model with experiment afterwards [9]

III. THE GROUND STATE

Let us study the ground state of the model under consideration, then properties of the excitations will follow naturally. In Eq.(12) the field coefficients h_i obey

$$2Gh_i - \Delta_i = \frac{\kappa}{8} t_{ijk} h_j h_k \tag{26}$$

with the totally symmetric coefficients t_{ijk} equal to zero except for $t_{uds} = 1$ and with order parameters $\Delta_i \neq 0 \quad (i = u, d, s)$.

A tadpole graphs calculation gives for the gap equations the following result

$$2h_i + \frac{3a\kappa}{8G^2}\left(h^{(2)}_{ab} - h^{(1)}_{ab}\right) A_{abc} h^{(1)}_{ci} = \frac{N_c}{2\pi^2} m_i J_0(m_i^2) \tag{27}$$

where the left hand side is the contribution from (25) and the right hand side is the contribution of the quark loop from (10), [8].

The second term on the left hand side of Eq.(27) is the correction resulting from the Gaussian integrals of the steepest descent method, comprising the effects of small fluctuations around the stationary path. If one puts for a moment $a = 0$ in Eq.(27), and combines the result with Eqs.(26), one finds gap equations which are very similar to the ones obtained in [4]. For this case one obtains values of (m_u, m_s) which are uniquely related to values of (G, κ).

The general case, which we have when $a > 0$ in Eq.(27), yields in turn a region for m_u, m_s, where three values of couplings (G, κ) are possible.

Conversely, one can study the solutions: $m_u = m_u(G, \kappa)$, $m_s = m_s(G, \kappa)$ at fixed values of input parameters: $\Lambda, \hat{m}_u = \hat{m}_d, \hat{m}_s$. Again we find several extremal solutions. The minima will be identified in a future analysis of the effective potential. We refer to [8] for graphical displays.

Multivalued solutions of the gap equations have been obtained in a different context in [10].

IV. CONCLUDING REMARKS

The purpose of this work has been twofold. Firstly we have developed the technique which is necessary to go beyound the lowest order SPA in the problem of the path integral bosonization of the 't Hooft six quark interaction. This technique is rather general and can be readily used in other applications. Second, we have explored with considerable detail the implications of taking the quantum fluctuations in account in the description of the hadronic vacuum. We encountered several classes of solutions. These multiple vacua may have very interesting physical consequences and applications.

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